

# Optimal control for a phase field system with a possibly singular potential

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## Abstract

In this paper we study a distributed control problem for a phase field system of Caginalp type with logarithmic potential. The main aim of this work would be to force the location of the diffuse interface to be as close as possible to a prescribed set. However, due to the discontinuous character of the cost functional, we have to approximate it by a regular one and, in this case, we solve the associated control problem and derive the related first order necessary optimality conditions.

**Key words:** Phase field system, phase transition, singular potentials, optimal control, optimality conditions, adjoint state system.

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# 1 Introduction

This paper is concerned with the study of a distributed control problem for a Caginalp type PDE system (cf. [6] and [5])

$$\partial_t \vartheta - \Delta \vartheta + \ell \partial_t \varphi = \sigma \quad \text{and} \quad \partial_t \varphi - \Delta \varphi + \mathcal{W}'(\varphi) = \ell \vartheta \quad \text{in } Q := (0, T) \times \Omega \quad (1.1)$$

where  $\Omega$  is the domain where the evolution takes place,  $T$  is some final time,  $\vartheta$  denotes the relative temperature around some critical value that is taken to be 0 without loss of generality, and  $\varphi$  is the order parameter. Moreover,  $\ell$  is a positive coefficient that is proportional to the latent heat, and  $\sigma$  is some source term. Finally,  $\mathcal{W}'$  represents the derivative of a double-well potential  $\mathcal{W}$ , and the typical example is the classical regular potential  $\mathcal{W}_{reg}$  defined by

$$\mathcal{W}_{reg}(r) = \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R}. \quad (1.2)$$

However, different choices of  $\mathcal{W}$  are possible, and a thermodynamically significant example is given by the so-called logarithmic double-well potential, namely

$$\mathcal{W}_{log}(r) = ((1+r) \ln(1+r) + (1-r) \ln(1-r)) - cr^2, \quad r \in (-1, 1) \quad (1.3)$$

where  $c > 0$  is large enough in order to kill convexity. More generally, the potential  $\mathcal{W}$  could be just the sum  $\mathcal{W} = \widehat{\beta} + \widehat{\pi}$ , where  $\widehat{\beta}$  is a convex function that is allowed to take the value  $+\infty$  somewhere, and  $\widehat{\pi}$  is a smooth perturbation (not necessarily concave). In such a case,  $\widehat{\beta}$  is supposed to be proper and lower semicontinuous so that its subdifferential is well-defined and can replace the derivative which might not exist. A typical example is the so-called double obstacle potential

$$\mathcal{W}_{obs}(r) = I_{[-1,1]}(r) - cr^2, \quad (1.4)$$

where  $I_{[-1,1]}$  denotes the indicator function of the set  $[-1, 1]$  which takes value 0 in  $[-1, 1]$  and  $+\infty$  outside. Of course, the second equation (1.1) becomes a differential inclusion.

The mathematical literature on (1.1) is rather vast and we confine ourselves to quote the pioneering paper [11] and the more recent ones [17], [12], [15] dealing respectively with the cases of regular, singular and non-smooth potentials.

Moreover, initial conditions like  $\vartheta(0) = \vartheta_0$  and  $\varphi(0) = \varphi_0$  and suitable boundary conditions must complement the above equations. As far as the latter are concerned, we take the homogeneous Dirichlet and Neumann boundary conditions, respectively, that is

$$\vartheta = 0 \quad \text{and} \quad \partial_n \varphi = 0 \quad \text{on } \Sigma := (0, T) \times \Gamma$$

where  $\Gamma$  is the boundary of  $\Omega$  and  $\partial_n$  is the (say, outward) normal derivative. We note that the latter is very common in the literature and that the former could be replaced by an inhomogeneous one.

The aim of this paper is to study a related optimal control problem, the control being associated to the forcing term  $\sigma$  that appears on the right-hand side of the first

equation (1.1). Namely, we take  $\sigma(t, x) = m(x)u(t, x)$ , where  $m$  is a given nonnegative function on  $\Omega$  and  $u$  is the control. Thus, the state system takes the following form

$$\partial_t \vartheta - \Delta \vartheta + \ell \partial_t \varphi = mu \quad \text{in } Q \quad (1.5)$$

$$\partial_t \varphi - \Delta \varphi + \beta(\varphi) + \pi(\varphi) \ni \ell \vartheta \quad \text{in } Q \quad (1.6)$$

$$\vartheta = 0 \quad \text{and} \quad \partial_n \varphi = 0 \quad \text{on } \Sigma \quad (1.7)$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0 \quad \text{on } \Omega \quad (1.8)$$

and the control  $u$  is supposed to vary in some control box  $\mathcal{U}_{ad}$ . We would like to force the location of the diffuse interface of  $\varphi$ , i.e., of the set  $E_\varepsilon(\varphi)$  where the state  $\varphi$  takes values between  $-\varepsilon$  and  $\varepsilon$ , for some given  $\varepsilon > 0$ , to be as close as possible to a prescribed set  $E \subset Q$ . To do that, by denoting by  $\chi_E$  the characteristic function of  $E$  and by  $g$  the characteristic function of the interval  $[-\varepsilon, \varepsilon]$ , we introduce the cost functional

$$\mathcal{J}_0(u) := \frac{1}{2} \int_Q (g(\varphi) - \chi_E)^2 \quad (1.9)$$

where  $(\vartheta, \varphi)$  is the state corresponding to  $u$ . More generally, we could take, e.g.,

$$\mathcal{J}(u) := \frac{1}{2} \int_Q (g(\varphi) - \chi_E)^2 + \frac{\kappa}{2} \int_Q (\vartheta - \vartheta_Q)^2 \quad (1.10)$$

where the desired temperature  $\vartheta_Q \in L^2(Q)$  and the constant  $\kappa \geq 0$  are given. In this case, the optimal control (if it exists) balances the closeness of  $E_\varepsilon(\varphi)$  to  $E$  and the smallness of the difference  $|\vartheta - \vartheta_Q|$ , depending on the value of the coefficient  $\kappa$ . However, such problems look difficult for every reasonable control box  $\mathcal{U}_{ad}$ . As this is mainly due to the discontinuous character of  $g$ , we replace the characteristic function  $g$  by a continuous approximation of it (still denoted by  $g$ ), and a possible choice is the following

$$g(r) := \frac{\lambda}{((r^2 - \varepsilon^2)^+)^2 + \lambda} \quad \text{for } r \in \mathbb{R}$$

where  $\lambda > 0$  is small. At this point, we can generalize the problem and allow  $g$  to be any continuous function on  $\mathbb{R}$  satisfying some growth condition that makes the cost functional meaningful for every admissible control  $u$ , and boundedness is surely suitable. Moreover, even  $\chi_E$  can be replaced by a more general given function.

Thus, the control problem we address in this paper consists in minimizing the cost functional

$$\mathcal{J}(u) := \frac{1}{2} \int_Q (g(\varphi) - \chi)^2 + \frac{\kappa}{2} \int_Q (\vartheta - \vartheta_Q)^2 \quad (1.11)$$

depending on the state variables  $\vartheta$  and  $\varphi$  satisfying the above state system, over all the controls belonging to some control box  $\mathcal{U}_{ad}$ , where  $\chi$  and  $\vartheta_Q$  are given in  $L^2(Q)$ ,  $\kappa$  is a nonnegative constant and  $g$  is a prescribed real function on  $\mathbb{R}$ , that we assume to be at least continuous and bounded. As far as the control box is concerned, we take

$$\mathcal{U}_{ad} := \{u \in L^2(Q) : u_{\min} \leq u \leq u_{\max} \text{ a.e. in } Q\} \quad (1.12)$$

where  $u_{\min}$  and  $u_{\max}$  are given bounded functions.

Let us mention here that in our approach the existence of an optimal control is proven for a quite general class of potentials  $\mathcal{W}$ : indeed,  $\mathcal{W}$  is assumed to be a smooth perturbation of a convex function  $\widehat{\beta}$  possibly taking the value  $+\infty$  somewhere. Notice that all the three examples (1.2), (1.3), and (1.4) fit these assumptions. However, we point out that the derivation of the first order necessary optimality conditions can be made only in case of regular (e.g. (1.2)) and singular (e.g. (1.3)) potentials (cf. Section 4). Hence, the main novelty of the present contribution consists in the fact that we can deal with quite general potentials  $\mathcal{W}$  (even singular) in the phase equation and quite general cost functions  $\mathcal{J}$ . Up to our knowledge, indeed, the literature on optimal control for Caginalp type phase field models is quite poor and often restricted to the case of regular potentials, or dealing with approximating problems when first order optimality conditions are discussed. In this framework, let us quote the papers [13, 14] and references therein and also [2, 3, 7, 8, 9, 18, 21, 23] for different types of phase field models.

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. The well-posedness of the state system and the existence of an optimal control will be shown in Sections 3 and 4, respectively, while the rest of the paper is devoted to the derivation of first order necessary conditions for optimality. The final result will be proved in Section 6 and it is prepared in Sections 5, which is devoted to the study of the control-to-state mapping.

## 2 Statement of the problem and results

In this section, we describe the problem under investigation and present our results. From now on, for simplicity and without any loss of generality we take  $\ell = 1$  in (1.5)–(1.8). As in the Introduction,  $\Omega$  is the body where the evolution takes place. We assume  $\Omega \subset \mathbb{R}^3$  to be open, bounded, connected, of class  $C^{1,1}$ , and we write  $|\Omega|$  for its Lebesgue measure. Moreover,  $\Gamma$  and  $\partial_n$  still stand for the boundary of  $\Omega$  and the outward normal derivative, respectively. Given a finite final time  $T > 0$ , we set for convenience

$$Q_t := (0, t) \times \Omega \quad \text{and} \quad \Sigma_t := (0, t) \times \Gamma \quad \text{for every } t \in (0, T] \quad (2.1)$$

$$Q := Q_T, \quad \text{and} \quad \Sigma := \Sigma_T. \quad (2.2)$$

Now, we specify the assumptions on the structure of our system. We assume that

$$m \in L^\infty(\Omega) \quad \text{and} \quad m \geq 0 \quad \text{a.e. in } \Omega \quad (2.3)$$

$$\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty] \quad \text{is convex, proper and l.s.c.} \quad \text{with} \quad \widehat{\beta}(0) = 0 \quad (2.4)$$

$$\widehat{\pi} : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is a } C^1 \text{ function} \quad \text{and} \quad \widehat{\pi}' \quad \text{is Lipschitz continuous.} \quad (2.5)$$

We set for convenience

$$\beta := \partial \widehat{\beta} \quad \text{and} \quad \pi := \widehat{\pi}' \quad (2.6)$$

and denote by  $D(\beta)$  and  $D(\widehat{\beta})$  the effective domains of  $\beta$  and  $\widehat{\beta}$ , respectively. Moreover,  $\beta_\varepsilon$  is the Yosida regularization of  $\beta$  at level  $\varepsilon$  and  $\beta^\circ(r)$  denotes the element of  $\beta(r)$  having minimum modulus for every  $r \in D(\beta)$  (see, e.g., [4, p. 28]). It is well known that both  $\beta$

and  $\beta_\varepsilon$  are maximal monotone operators and that  $\beta_\varepsilon$  is even single-valued and Lipschitz continuous. Furthermore (see, e.g., [4, Prop. 2.6, p. 28]), we have

$$|\beta_\varepsilon(r)| \leq |\beta^\circ(r)| \quad \text{and} \quad \beta_\varepsilon(r) \rightarrow \beta^\circ(r) \quad \text{for } r \in D(\beta). \quad (2.7)$$

Next, in order to simplify notations, we set

$$V := H^1(\Omega), \quad V_0 := H_0^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0\} \quad (2.8)$$

and endow these spaces with their natural norms. The symbol  $\|\cdot\|_X$  stands for the norm in the generic Banach space  $X$ , while  $\|\cdot\|_p$  is the usual norm in both  $L^p(\Omega)$  and  $L^p(Q)$ , for  $1 \leq p \leq \infty$ . Finally, for  $v \in L^2(0, T; X)$  the function  $1 * v$  is defined by

$$(1 * v)(t) := \int_0^t v(s) ds \quad \text{for } t \in [0, T] \quad (2.9)$$

(note that the symbol  $*$  is usually employed for convolution products).

At this point, we describe the state system. Given  $\vartheta_0$  and  $\varphi_0$  such that

$$\vartheta_0 \in V_0 \quad (2.10)$$

$$\varphi_0 \in V \quad \text{and} \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega) \quad (2.11)$$

we look for a triplet  $(\vartheta, \varphi, \xi)$  satisfying

$$\vartheta \in H^1(0, T; H) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)) \quad (2.12)$$

$$\varphi \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.13)$$

$$\xi \in L^2(0, T; H) \quad (2.14)$$

$$\partial_t \vartheta - \Delta \vartheta + \partial_t \varphi = mu \quad \text{a.e. in } Q \quad (2.15)$$

$$\partial_t \varphi - \Delta \varphi + \xi + \pi(\varphi) = \vartheta \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q \quad (2.16)$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega. \quad (2.17)$$

Our first result, whose proof is sketched in Section 3, ensures well-posedness with the prescribed regularity, stability and continuous dependence on the control variable in suitable topologies.

**Theorem 2.1.** *Assume (2.3)–(2.5) and (2.10)–(2.11). Then, for every  $u \in L^2(Q)$ , problem (2.15)–(2.17) has a unique solution  $(\vartheta, \varphi, \xi)$  satisfying (2.12)–(2.14), and the estimate*

$$\begin{aligned} & \|\vartheta\|_{H^1(0, T; H) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega))} \\ & + \|\varphi\|_{H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} + \|\xi\|_{L^2(0, T; H)} \leq C_1 \end{aligned} \quad (2.18)$$

holds true for some constant  $C_1$  that depends only on  $\Omega$ ,  $T$ , the structure (2.3)–(2.5) of the system, the norms of the initial data associated to (2.10)–(2.11) and  $\|u\|_2$ . Moreover, if  $u_i \in L^2(Q)$ ,  $i = 1, 2$ , are given and  $(\vartheta_i, \varphi_i, \xi_i)$  are the corresponding solutions, then the estimate

$$\begin{aligned} & \|\vartheta_1 - \vartheta_2\|_{L^2(0, T; H)} + \|(1 * \vartheta_1) - (1 * \vartheta_2)\|_{L^\infty(0, T; V_0)} \\ & + \|\varphi_1 - \varphi_2\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} \\ & \leq C' \|(1 * u_1) - (1 * u_2)\|_{L^2(0, T; H)} \leq C'' \|u_1 - u_2\|_{L^2(0, T; H)} \end{aligned} \quad (2.19)$$

holds true with constants  $C'$  and  $C''$  that depend only on  $\Omega$ ,  $T$ ,  $\pi$  and  $m$ .

Some further regularity of the solution is stated in the next result, whose proof is given in Section 3.

**Theorem 2.2.** *The following properties hold true.*

i) Assume (2.3)–(2.5) and (2.10)–(2.11). Moreover, let

$$\varphi_0 \in W \quad \text{and} \quad \beta^\circ(\varphi_0) \in H. \quad (2.20)$$

Then, the unique solution  $(\vartheta, \varphi, \xi)$  given by Theorem 2.1 also satisfies

$$\varphi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad (2.21)$$

$$\xi \in L^\infty(0, T; H) \quad (2.22)$$

$$\|\varphi\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)} + \|\xi\|_{L^\infty(0,T;H)} \leq C_2 \quad (2.23)$$

$$\varphi \in C^0(\overline{Q}) \quad \text{and} \quad \|\varphi\|_\infty \leq C_2 \quad (2.24)$$

for some constant  $C_2$  that depends only on  $\Omega$ ,  $T$ , the structure (2.3)–(2.5) of the system, the norms of the initial data associated to (2.10)–(2.11), (2.20) and  $\|u\|_2$ .

ii) If in addition  $\vartheta_0 \in L^\infty(\Omega)$  and  $u \in L^\infty(0, T; H)$ , we also have

$$\vartheta \in L^\infty(Q) \quad \text{and} \quad \|\vartheta\|_\infty \leq C_3 \quad (2.25)$$

with a similar constant  $C_3$  that depends on  $\|\vartheta_0\|_\infty$  and  $\|u\|_{L^\infty(0,T;H)}$  as well.

iii) By further assuming  $\beta^\circ(\varphi_0) \in L^\infty(\Omega)$ , we have that  $\xi \in L^\infty(Q)$  and

$$\|\xi\|_{L^\infty(Q)} \leq C_4 \quad (2.26)$$

with a constant  $C_4$  that depends on  $C_3$  and  $\|\beta^\circ(\varphi_0)\|_\infty$  in addition.

The well-posedness result for problem (2.15)–(2.17) given by Theorem 2.1 allows us to introduce the control-to-state mapping  $\mathcal{S}$  and to address the corresponding control problem. We define

$$\mathcal{X} := L^\infty(Q) \quad (2.27)$$

$$\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2 \quad \text{where} \quad (2.28)$$

$$\mathcal{Y}_1 := \{v \in L^2(Q) : 1 * v \in L^2(0, T; V_0)\} \quad (2.29)$$

$$\mathcal{Y}_2 := L^\infty(0, T; H) \cap L^2(0, T; V) \quad (2.30)$$

$$\begin{aligned} \mathcal{S} : \mathcal{X} &\rightarrow \mathcal{Y}, \quad u \mapsto \mathcal{S}(u) =: (\vartheta, \varphi) \quad \text{where} \\ (\vartheta, \varphi, \xi) &\text{ is the unique solution to (2.12)–(2.17) corresponding to } u. \end{aligned} \quad (2.31)$$

Next, in order to introduce the control box and the cost functional, we assume that

$$u_{\min}, u_{\max} \in L^\infty(Q) \quad \text{satisfy} \quad u_{\min} \leq u_{\max} \quad \text{a.e. in } Q \quad (2.32)$$

$$g : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is continuous and bounded} \quad (2.33)$$

$$\kappa \in [0, +\infty) \quad \text{and} \quad \chi, \vartheta_Q \in L^2(Q) \quad (2.34)$$

and define  $\mathcal{U}_{ad}$  and  $\mathcal{J}$  according to the Introduction. Namely, we set

$$\mathcal{U}_{ad} := \{u \in \mathcal{X} : u_{\min} \leq u \leq u_{\max} \text{ a.e. in } Q\} \quad (2.35)$$

$$\begin{aligned} \mathcal{J} &:= \mathcal{F} \circ \mathcal{S} : \mathcal{X} \rightarrow \mathbb{R} \quad \text{where } \mathcal{F} : \mathcal{Y} \rightarrow \mathbb{R} \text{ is defined by} \\ \mathcal{F}(\vartheta, \varphi) &:= \frac{1}{2} \int_Q (g(\varphi) - \chi)^2 + \frac{\kappa}{2} \int_Q (\vartheta - \vartheta_Q)^2. \end{aligned} \quad (2.36)$$

Here is our first result on the control problem; for the proof we refer to Section 4.

**Theorem 2.3.** *Assume (2.3)–(2.5) and (2.10)–(2.11), and let  $\mathcal{U}_{ad}$  and  $\mathcal{J}$  be defined by (2.35)–(2.36). Then, there exists  $u^* \in \mathcal{U}_{ad}$  such that*

$$\mathcal{J}(u^*) \leq \mathcal{J}(u) \quad \text{for every } u \in \mathcal{U}_{ad}. \quad (2.37)$$

From now on, it is understood that the assumptions (2.3)–(2.5) and those on the structure and on the initial data are satisfied and that the map  $\mathcal{S}$ , the cost functionals  $\mathcal{F}$  and  $\mathcal{J}$  and the control box  $\mathcal{U}_{ad}$  are defined in (2.27)–(2.36). Thus, we do not remind anything of that in the statements given in the sequel.

Our next aim is to formulate the first order necessary optimality conditions. As  $\mathcal{U}_{ad}$  is convex, the desired necessary condition for optimality is

$$\langle D\mathcal{J}(u^*), u - u^* \rangle \geq 0 \quad \text{for every } u \in \mathcal{U}_{ad} \quad (2.38)$$

provided that the derivative  $D\mathcal{J}(u^*)$  exists in the dual space  $\mathcal{X}^*$  at least in the Gâteaux sense. Then, the natural approach consists in proving that  $\mathcal{S}$  is Fréchet differentiable at  $u^*$  and applying the chain rule to  $\mathcal{J} = \mathcal{F} \circ \mathcal{S}$ . We can properly tackle this project under further assumptions on the nonlinearities  $\beta$ ,  $\pi$  and  $g$ . Namely, we also suppose that

$$D(\beta) \text{ is an open interval and } \beta \text{ is a single-valued on } D(\beta) \quad (2.39)$$

$$\beta \text{ and } \pi \text{ are } C^2 \text{ functions and } g \text{ is a } C^1 \text{ function} \quad (2.40)$$

and observe that, in particular,  $\beta^\circ = \beta$ .

We remark that both the regular potential (1.2) and the logarithmic potential (1.3) satisfy the above assumptions on  $\beta$  and  $\pi$ . Another possible choice of  $\beta$  is given by

$$\beta(r) := 1 - \frac{1}{r+1} \quad \text{for } r > -1 \quad (2.41)$$

and it corresponds to  $\widehat{\beta}$  defined by

$$\widehat{\beta}(r) := r - \ln(r+1) \quad \text{if } r > -1 \quad \text{and} \quad \widehat{\beta}(r) := +\infty \quad \text{otherwise} \quad (2.42)$$

with  $\widehat{\beta}$  taking the minimum 0 at 0, as required by assumption (2.4). Such an operator  $\beta$  yields an example of a different behavior for negative and positive values, singular near  $-1$  and with a somehow linear growth at  $+\infty$ .

Furthermore, we notice that the inclusion in (2.16) becomes  $\xi = \beta(\varphi)$  and that  $\beta$  and  $\pi$  enter the problem through their sum, mainly. Hence, we set for brevity

$$\gamma := \beta + \pi \quad (2.43)$$

and observe that  $\gamma$  is a  $C^2$  function on  $D(\beta)$ .

Since assumptions (2.39)–(2.40) force  $\beta(r)$  to tend to  $\pm\infty$  as  $r$  tends to a finite end-point of  $D(\beta)$ , if any, we see that combining the further requirement (2.39)–(2.40) with the boundedness of  $\varphi$  and  $\xi$  given by Theorem 2.2 immediately yields

**Corollary 2.4.** *Under all the assumptions of Theorem 2.2, suppose that (2.39)–(2.40) hold, in addition. Then, the component  $\varphi$  of the solution  $(\vartheta, \varphi, \xi)$  also satisfies*

$$\varphi_\bullet \leq \varphi \leq \varphi^\bullet \quad \text{in } \overline{Q} \quad (2.44)$$

for some constants  $\varphi_\bullet, \varphi^\bullet \in D(\beta)$  that depend only on  $\Omega, T$ , the structure (2.3)–(2.5) and (2.39)–(2.40) of the system, the norms of the initial data associated to (2.10)–(2.11), and the norms  $\|u\|_\infty, \|\vartheta_0\|_\infty$  and  $\|\beta(\varphi_0)\|_\infty$ .

As we shall see in Section 5, the computation of the Fréchet derivative of  $\mathcal{S}$  leads to the linearized problem that we describe at once and that can be stated starting from a generic element  $\bar{u} \in \mathcal{X}$ . Let  $\bar{u} \in \mathcal{X}$  and  $h \in \mathcal{X}$  be given. We set  $(\bar{\vartheta}, \bar{\varphi}) := \mathcal{S}(\bar{u})$ . Then the linearized problem consists in finding  $(\Theta, \Phi)$  satisfying

$$\Theta \in H^1(0, T; H) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)) \quad (2.45)$$

$$\Phi \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.46)$$

and solving the following problem

$$\partial_t \Theta - \Delta \Theta + \partial_t \Phi = mh \quad \text{a.e. in } Q \quad (2.47)$$

$$\partial_t \Phi - \Delta \Phi + \gamma'(\bar{\varphi}) \Phi = \Theta \quad \text{a.e. in } Q \quad (2.48)$$

$$\Theta(0) = 0 \quad \text{and} \quad \Phi(0) = 0 \quad \text{a.e. in } \Omega. \quad (2.49)$$

**Proposition 2.5.** *Let  $\bar{u} \in \mathcal{X}$  and  $(\bar{\vartheta}, \bar{\varphi}) = \mathcal{S}(\bar{u})$ . Then, for every  $h \in \mathcal{X}$ , there exists a unique pair  $(\Theta, \Phi)$  satisfying (2.45)–(2.46) and solving the linearized problem (2.47)–(2.49). Moreover, the inequality*

$$\|(\Theta, \Phi)\|_{\mathcal{Y}} \leq C_5 \|h\|_{\mathcal{X}} \quad (2.50)$$

holds true with a constant  $C_5$  that depend only on  $\Omega, T$ , the structure (2.3)–(2.5) and (2.39)–(2.40) of the system, the norms of the initial data associated to (2.10)–(2.11), and the norms  $\|\bar{u}\|_\infty, \|\vartheta_0\|_\infty$  and  $\|\beta(\varphi_0)\|_\infty$ . In particular, the linear map  $\mathcal{D} : h \mapsto (\Theta, \Phi)$  is continuous from  $\mathcal{X}$  to  $\mathcal{Y}$ .

Namely, we shall prove that the Fréchet derivative  $D\mathcal{S}(\bar{u}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  actually exists and coincides with the map  $\mathcal{D}$  introduced in the last statement. This will be done in



Section 5. Once this is established, we may use the chain rule with  $\bar{u} := u^*$  to prove that the necessary condition (2.38) for optimality takes the form

$$\int_Q (g(\varphi^*) - \chi) g'(\varphi^*) \Phi + \kappa \int_Q (\vartheta^* - \vartheta_Q) \Theta \geq 0 \quad \text{for any } u \in \mathcal{U}_{ad}, \quad (2.51)$$

where  $(\vartheta^*, \varphi^*) = \mathcal{S}(u^*)$  and, for any given  $u \in \mathcal{U}_{ad}$ , the pair  $(\Theta, \Phi)$  is the solution to the linearized problem corresponding to  $h = u - u^*$ .

The final step then consists in eliminating the pair  $(\Theta, \Phi)$  from (2.51). This will be done by introducing a pair  $(p, q)$  that fulfills the regularity requirements

$$p \in H^1(0, T; H) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)) \quad (2.52)$$

$$q \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.53)$$

and solves the following adjoint system:

$$-\partial_t p - \Delta p - q = \kappa(\vartheta^* - \vartheta_Q) \quad \text{a.e. in } Q \quad (2.54)$$

$$-\partial_t q - \Delta q + \gamma'(\varphi^*) q - \partial_t p = (g(\varphi^*) - \chi) g'(\varphi^*) \quad \text{a.e. in } Q \quad (2.55)$$

$$p(T) = q(T) = 0 \quad \text{a.e. in } \Omega. \quad (2.56)$$

Here, let us recall (2.8) and note that, as in previous cases (cf. (2.12)–(2.17) and (2.45)–(2.49)), the Dirichlet boundary condition for  $p$  is contained in (2.52) whereas the Neumann boundary condition for  $q$  is in (2.53).

**Theorem 2.6.** *Let  $u^*$  and  $(\vartheta^*, \varphi^*) = \mathcal{S}(u^*)$  be an optimal control and the corresponding state. Then the adjoint problem (2.54)–(2.56) has a unique solution  $(p, q)$  satisfying the regularity conditions (2.52)–(2.53).*

Our last result establishes optimality conditions.

**Theorem 2.7.** *Let  $u^*$  be an optimal control. Moreover, let  $(\vartheta^*, \varphi^*) = \mathcal{S}(u^*)$  and  $(p, q)$  be the associate state and the unique solution to the adjoint problem (2.54)–(2.56) given by Theorem 2.6. Then we have*

$$m(x) p(t, x) (u - u^*(t, x)) \geq 0 \quad \text{for every } u \in [u_{\min}(t, x), u_{\max}(t, x)], \\ \text{for a.a. } (t, x) \in Q. \quad (2.57)$$

In particular,  $mp = 0$  in the subset of  $Q$  where  $u_{\min} < u^* < u_{\max}$ .

A straightforward consequence of Theorem 2.7 is here stated.

**Corollary 2.8.** *Under the conditions of Theorem 2.7, the optimal control  $u^*$  reads*

$$u^* = \begin{cases} u_{\min} & \text{a.e. on the set } \{(t, x) : p(t, x) > 0 \text{ and } m(x) > 0\} \\ u_{\max} & \text{a.e. on the set } \{(t, x) : p(t, x) < 0 \text{ and } m(x) > 0\} \\ \text{undetermined} & \text{elsewhere.} \end{cases}$$

In the remainder of the paper, we often owe to the Hölder inequality and to the elementary Young inequalities

$$ab \leq \alpha a^{1/\alpha} + (1 - \alpha) b^{1/(1-\alpha)} \quad \text{and} \quad ab \leq \delta a^2 + \frac{1}{4\delta} b^2$$

for every  $a, b \geq 0$ ,  $\alpha \in (0, 1)$  and  $\delta > 0$  (2.58)

in performing our a priori estimates. To this regard, in order to avoid a boring notation, we use the following general rule to denote constants. The small-case symbol  $c$  stands for different constants which depend only on  $\Omega$ , on the final time  $T$ , the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. A small-case  $c$  with a subscript like  $c_\delta$  indicates that the constant might depend on the parameter  $\delta$ , in addition. Hence, the meaning of  $c$  and  $c_\delta$  might change from line to line and even in the same chain of equalities or inequalities. On the contrary, different symbols (e.g., capital letters) stand for precise constants which we can refer to.

### 3 The state system

This section is devoted to the proofs of Theorems 2.1 and 2.2. As far as the former is concerned, we notice that the initial-boundary value problem under study is a quite standard phase field system and that a number of results on it can be found in the literature (see, e.g., [5, 10, 20], and references therein). Nevertheless, we prefer to sketch the basic a priori estimates that correspond to the regularity (2.12)–(2.14) of the solution and to estimate (2.18), for the reader convenience. A complete existence proof can be obtained by regularizing the problem, performing the same estimates on the corresponding solution, and passing to the limit through compactness results. We also give a short proof of (2.19) (whence uniqueness follows as a consequence) and conclude the discussion on Theorem 2.1.

As said, we derive just formal a priori estimates. We multiply (2.15) by  $\vartheta$ ; then we add  $\varphi$  to both sides of (2.16) and test by  $\partial_t \varphi$ ; finally, we sum up and integrate over  $Q_t$  with  $t \in (0, T)$ . As the terms involving the product  $\vartheta \partial_t \varphi$  cancel out, by exploiting a standard chain rule for subdifferentials (see, e.g., [4, Lemme 3.3, p. 73]) we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\vartheta(t)|^2 + \int_{Q_t} |\nabla \vartheta|^2 + \int_{Q_t} |\partial_t \varphi|^2 + \frac{1}{2} \|\varphi(t)\|_V^2 + \frac{1}{2} \int_{\Omega} \widehat{\beta}(\varphi(t)) \\ &= \frac{1}{2} \int_{\Omega} |\vartheta_0|^2 + \frac{1}{2} \|\varphi_0\|_V^2 + \frac{1}{2} \int_{\Omega} \widehat{\beta}(\varphi_0) + \int_{Q_t} m u \vartheta + \int_{Q_t} (\varphi - \pi(\varphi)) \partial_t \varphi. \end{aligned} \quad (3.1)$$

The last integral on the left-hand side is nonnegative thanks to (2.4) and the first three terms on the right-hand side are under control, due to (2.10)–(2.11). Since (cf. (2.5)–(2.6))  $|\varphi - \pi(\varphi)| \leq c(|\varphi| + 1)$  and (2.3) holds, the last two terms on the right-hand side of (3.1) can be easily dealt with by the Young inequality and the Gronwall lemma. Then, we deduce the estimate

$$\|\vartheta\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\varphi\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c. \quad (3.2)$$

Since  $\partial_t \varphi$  is by now bounded in  $L^2(Q)$ , we can test (2.15) by  $\partial_t \vartheta$  in order to infer that

$$\int_{Q_t} |\partial_t \vartheta|^2 + \frac{1}{2} \int_{\Omega} |\nabla \vartheta(t)|^2 = \frac{1}{2} \int_{\Omega} |\nabla \vartheta_0|^2 + \int_{Q_t} (mu - \partial_t \varphi) \partial_t \vartheta.$$

Thus, (2.10) and the Young inequality enable us to recover

$$\|\vartheta\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c \quad (3.3)$$

as well. At this point, owing to (3.2)–(3.3),  $\Delta \vartheta$  and  $-\Delta \varphi + \xi$  are bounded in  $L^2(0, T; H)$ , as one clearly sees from equations (2.15)–(2.16). Hence, a standard monotonicity argument (test some regularization of (2.16) by the analogue of  $\xi = \beta(\varphi)$ ) yields that both  $\Delta \varphi$  and  $\xi$  are bounded in  $L^2(0, T; H)$ . Then, elliptic regularity allows us to derive the complete estimate (2.18).

Let us pass to (2.19). We first integrate (2.15) with respect to time and get the equation

$$\vartheta - \Delta(1 * \vartheta) + \varphi = \vartheta_0 + \varphi_0 + m(1 * u). \quad (3.4)$$

Now, we fix  $u_i \in L^2(Q)$ ,  $i = 1, 2$ , and consider two corresponding solutions  $(\vartheta_i, \varphi_i, \xi_i)$  with the same initial data. We write (3.4) for both of them and multiply the difference by  $\vartheta := \vartheta_1 - \vartheta_2$ . At the same time, we write (2.16) for both solution and multiply the difference by  $\varphi$ , where  $\varphi := \varphi_1 - \varphi_2$ . Then, we add the equalities we obtain to each other and integrate over  $Q_t$ . The terms involving the product  $\varphi \vartheta$  cancel out. Hence, by also setting  $u := u_1 - u_2$  and  $\xi := \xi_1 - \xi_2$  for brevity, we have

$$\begin{aligned} & \int_{Q_t} |\vartheta|^2 + \frac{1}{2} \int_{\Omega} |\nabla(1 * \vartheta)(t)|^2 + \frac{1}{2} \int_{\Omega} |\varphi(t)|^2 + \int_{Q_t} |\nabla \varphi|^2 + \int_{Q_t} \xi \varphi \\ &= \int_{Q_t} m(1 * u) \vartheta - \int_{Q_t} (\pi(\varphi_1) - \pi(\varphi_2)) \varphi \\ &\leq c \|1 * u\|_{L^2(Q)}^2 + \frac{1}{2} \int_{Q_t} |\vartheta|^2 + c \int_{Q_t} |\varphi|^2 \end{aligned}$$

where we used the boundedness of  $m$  and the Lipschitz continuity of  $\pi$  (see (2.3) and (2.5)–(2.6) once more). As the last integral on the left-hand side is nonnegative since  $\beta$  is monotone, we obtain the desired estimate (2.19) just by rearranging and applying the Gronwall lemma.  $\square$

Now, we prove Theorem 2.2 using the same strategy of a formal argumentation. First, we consider the equation obtained by differentiating (2.16) with respect to time and test it by  $\partial_t \varphi$ . Then, we have

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\partial_t \varphi(t)|^2 + \int_{Q_t} |\nabla \partial_t \varphi|^2 + \int_{Q_t} \beta'(\varphi) |\partial_t \varphi|^2 \\ &\leq \frac{1}{2} \int_{\Omega} |\partial_t \varphi(0)|^2 + \int_{Q_t} (\partial_t \vartheta - \pi'(\varphi) \partial_t \varphi) \partial_t \varphi. \end{aligned}$$

The monotonicity of  $\beta$  implies that the last term on the left-hand side is nonnegative; on the right-hand side, the last integral is already bounded thanks to (3.2)–(3.3) and to the

boundedness of  $\pi'$  (see (2.5)–(2.6)). Thus, just the norm of  $\partial_t \varphi(0)$  in  $L^2(\Omega)$  should be estimated, and this can be performed by recovering  $\partial_t \varphi(0)$  from equation (2.16) and then exploiting (2.10)–(2.11) as well as (2.20). Consequently, we obtain

$$\|\varphi\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V)} \leq c$$

and, in addition, the boundedness of  $-\Delta \varphi + \xi$  in  $L^\infty(0,T;H)$ . Now, it is straightforward to infer that the two separate terms  $\Delta \varphi$  and  $\xi$  are both bounded in  $L^\infty(0,T;H)$ . Then, the properties (2.21)–(2.23) follow; moreover, they imply that  $\varphi$  is bounded in  $C^0([0,T];C^0(\overline{\Omega})) = C^0(\overline{Q})$  since  $W$  is compactly embedded in  $C^0(\overline{\Omega})$  (see, e.g., [22, Sect. 8, Cor. 4]). This proves *i*). For the second statement *ii*), we observe that  $\vartheta$  turns out to be bounded whenever its initial value is bounded. Indeed, (2.16) can be written in the form

$$\partial_t \vartheta - \Delta \vartheta = mu - \partial_t \varphi \in L^\infty(0,T;H)$$

whence it suffices to apply, e.g., [16, Thm. 7.1, p. 181] with  $r = \infty$  and  $q = 2$ . Finally, we prove *iii*) by writing (2.15) in the form

$$\partial_t \varphi - \Delta \varphi + \xi = f := \vartheta - \pi(\varphi) \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q \quad (3.5)$$

and observing that  $f$  is bounded in  $L^\infty(Q)$  on account of the result *ii*) just proved. Now, we approximate  $\varphi$  by the solution  $\varphi_\varepsilon$  to the initial–boundary value problem obtained by keeping the same initial and boundary conditions and replacing (3.5) by

$$\partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon + \xi_\varepsilon = f := \vartheta - \pi(\varphi) \quad \text{and} \quad \xi_\varepsilon := \beta_\varepsilon(\varphi_\varepsilon) \quad \text{a.e. in } Q \quad (3.6)$$

where  $\beta_\varepsilon$  is the Yosida regularization of  $\beta$  at level  $\varepsilon > 0$ . Indeed, a standard argument shows that  $\varphi_\varepsilon$  converges to  $\varphi$  in the proper topology as  $\varepsilon$  tends to zero, so that *iii*) immediately follows whenever we prove that  $\xi_\varepsilon$  is bounded in  $L^\infty(Q)$  uniformly with respect to  $\varepsilon$ . To this end, by extending the sign function by  $\text{sign}(0) = 0$ , we notice that  $\text{sign } \beta_\varepsilon(r) = \text{sign } r$  for every  $r \in \mathbb{R}$  since  $\beta(0) \ni 0$  (see (2.4)) and set

$$\widehat{\beta}_{\varepsilon,p}(r) := \int_0^r |\beta_\varepsilon(s)|^{p-1} \text{sign } s \, ds \quad \text{for } r \in \mathbb{R} \text{ and } p > 1.$$

We obtain a nonnegative function. Then, we multiply (3.6) by  $|\xi_\varepsilon|^{p-1} \text{sign } \xi_\varepsilon$ , where  $p > 2$  is arbitrary, and integrate over  $Q_t$ . We have

$$\begin{aligned} & \int_\Omega \widehat{\beta}_{\varepsilon,p}(\varphi_\varepsilon(t)) + (p-1) \int_{Q_t} |\xi_\varepsilon|^{p-2} \beta'_\varepsilon(\varphi_\varepsilon) |\nabla \varphi_\varepsilon|^2 + \int_{Q_t} |\xi_\varepsilon|^p \\ &= \int_\Omega \widehat{\beta}_{\varepsilon,p}(\varphi_0) + \int_{Q_t} f |\xi_\varepsilon|^{p-1} \text{sign } \xi_\varepsilon. \end{aligned}$$

By noting that the first two terms on the left-hand side are nonnegative and owing to the Young inequality, we deduce that

$$\int_{Q_t} |\xi_\varepsilon|^p \leq \int_\Omega \widehat{\beta}_{\varepsilon,p}(\varphi_0) + \int_{Q_t} |f| |\xi_\varepsilon|^{p-1} \leq \int_\Omega \widehat{\beta}_{\varepsilon,p}(\varphi_0) + \frac{1}{p} \int_{Q_t} |f|^p + \frac{1}{p'} \int_{Q_t} |\xi_\varepsilon|^p.$$

By rearranging, we obtain

$$\int_{Q_t} |\xi_\varepsilon|^p \leq p \int_{\Omega} \widehat{\beta}_{\varepsilon,p}(\varphi_0) + \int_{Q_t} |f|^p$$

whence also (since  $(a+b)^\alpha \leq a^\alpha + b^\alpha$  for every  $a, b \geq 0$  and  $\alpha \in (0, 1)$ )

$$\|\xi_\varepsilon\|_{L^p(Q)} \leq p^{1/p} \left( \int_{\Omega} \widehat{\beta}_{\varepsilon,p}(\varphi_0) \right)^{1/p} + (|\Omega|T)^{1/p} \|f\|_\infty.$$

By letting  $p$  tend to infinity, we conclude that

$$\|\xi_\varepsilon\|_\infty \leq C_0 + \|f\|_\infty \quad \text{provided that} \quad \int_{\Omega} \widehat{\beta}_{\varepsilon,p}(\varphi_0) \leq C_0^p$$

and we just have to show that such a finite  $C_0$  exists. To this aim, we notice that  $r, \beta_\varepsilon(r)$  and  $\beta^\circ(r)$  have the same sign for every  $r \in \mathbb{R}$ ; on the other hand, (2.7) holds and even  $\beta^\circ$  is monotone. Hence, we have

$$\widehat{\beta}_{\varepsilon,p}(\varphi_0) = \left| \int_0^{\varphi_0} |\beta_\varepsilon(s)|^{p-1} ds \right| \leq |\varphi_0| |\beta^\circ(\varphi_0)|^{p-1} \quad \text{a.e. in } \Omega.$$

As both  $\varphi_0$  and  $\beta^\circ(\varphi_0)$  are bounded, the former since  $\varphi_0 \in W \subset L^\infty(\Omega)$  and the latter by assumption, we deduce that

$$\int_{\Omega} \widehat{\beta}_{\varepsilon,p}(\varphi_0) \leq |\Omega| \|\varphi_0\|_\infty \|\beta^\circ(\varphi_0)\|_\infty^{p-1} \leq C_0^p$$

with an obvious choice of  $C_0$ , and the proof is complete.  $\square$

## 4 Existence of an optimal control

The following section is devoted to the proof of Theorem 2.3. We use the direct method, observing first that  $\mathcal{U}_{ad}$  is nonempty. Then, we let  $\{u_n\}$  be a minimizing sequence for the optimization problem and, for any  $n$ , we take the corresponding solution  $(\varphi_n, \vartheta_n, \xi_n)$  to problem (2.15)–(2.17). Then,  $\{u_n\}$  is bounded in  $L^\infty(\Omega)$  and estimate (2.18) holds for  $(\varphi_n, \vartheta_n, \xi_n)$ . Therefore, we have for a subsequence

$$\begin{aligned} u_n &\rightarrow u && \text{weakly star in } L^\infty(\Omega) \\ \vartheta_n &\rightarrow \vartheta && \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)) \\ \varphi_n &\rightarrow \varphi && \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \\ \xi_n &\rightarrow \xi && \text{weakly star in } L^\infty(0, T; H). \end{aligned}$$

Then,  $u \in \mathcal{U}_{ad}$  since  $\mathcal{U}_{ad}$  is closed in  $\mathcal{X}$ , the initial conditions for  $\vartheta$  and  $\varphi$  are satisfied, and we can easily conclude by standard argument. Very shortly,  $\{\varphi_n\}$  converges strongly, e.g., in  $L^2(Q)$  and a.e. in  $Q$  (for a subsequence) by the Aubin-Lions compactness lemma (see, e.g., [19, Thm. 5.1, p. 58]), whence  $\pi(\varphi_n)$  converges to  $\pi(\varphi)$  in the same topology and  $\xi \in \beta(\varphi)$  (see, e.g., [1, Lemma 1.3, p. 42]). Thus,  $(\vartheta, \varphi, \psi)$  satisfies the variational formulation in the integral form of problem (2.15)–(2.17). On the other hand,  $\mathcal{F}(\vartheta_n, \varphi_n)$  converges both to the infimum of  $\mathcal{J}$  and to  $\mathcal{F}(\vartheta, \varphi)$ , since  $g(\varphi_n)$  converges to  $g(\varphi)$  a.e. in  $Q$  and it is bounded in  $L^\infty(Q)$  (see (2.33)). Therefore,  $u$  is an optimal control.  $\square$

## 5 The control-to-state mapping

As sketched in Section 2, the main point is the Fréchet differentiability of the control-to-state mapping  $\mathcal{S}$ . This involves the linearized problem (2.47)–(2.49), whose well-posedness is stated in Proposition 2.5. Thus, we first prove such a result.

As one can easily see by going through the proof of estimates (2.18) and (2.19) given in Section 3, what is stated in Theorem 2.1 can be extended to the problem obtained by replacing equation (2.16) of (2.15)–(2.17) by the more general one

$$\partial_t \varphi - \Delta \varphi + \xi + \alpha \pi(\varphi) = \vartheta \quad \text{and} \quad \xi \in \beta(\varphi)$$

where  $\alpha \in L^\infty(Q)$  is prescribed. Therefore, Proposition 2.5 follows as a trivial particular case. Namely, one just chooses  $\beta = 0$ ,  $\pi(r) = r$  and  $\alpha = \gamma'(\bar{\varphi})$  where  $\gamma$  is defined by (2.43) by starting from the original  $\beta$  and  $\pi$ . Indeed,  $\gamma'(\bar{\varphi})$  is bounded thanks to Corollary 2.4. In fact, estimate (2.50) holds more generally with  $\|h\|_{L^2(Q)}$  on the right-hand side.  $\square$

Here is the main result of this section.

**Theorem 5.1.** *Let  $\bar{u} \in \mathcal{X}$  and  $(\bar{\vartheta}, \bar{\varphi}) = \mathcal{S}(\bar{u})$ . Then,  $\mathcal{S}$  is Fréchet differentiable at  $(\bar{\vartheta}, \bar{\varphi})$  and the Fréchet derivative  $[D\mathcal{S}](\bar{\vartheta}, \bar{\varphi})$  precisely is the map  $\mathcal{D} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  defined in the statement of Proposition 2.5.*

*Proof.* We fix  $\bar{u} \in \mathcal{X}$  and the corresponding state  $(\bar{\vartheta}, \bar{\varphi})$  and, for  $h \in \mathcal{X}$  with  $\|h\|_{\mathcal{X}} \leq \Lambda$ , for some positive constant  $\Lambda$ , we set

$$(\vartheta^h, \varphi^h) := \mathcal{S}(\bar{u} + h) \quad \text{and} \quad (\zeta^h, \eta^h) := (\vartheta^h - \bar{\vartheta} - \Theta, \varphi^h - \bar{\varphi} - \Phi)$$

where  $(\Theta, \Phi)$  is the solution to the linearized problem corresponding to  $h$ . We have to prove that  $\|(\zeta^h, \eta^h)\|_{\mathcal{Y}}/\|h\|_{\mathcal{X}}$  tends to zero as  $\|h\|_{\mathcal{X}}$  tends to zero. More precisely, we show that

$$\|(\zeta^h, \eta^h)\|_{\mathcal{Y}} \leq c \|h\|_{L^2(Q)}^2 \quad (5.1)$$

for some constant  $c$ , and this is even stronger than necessary. First of all, we fix one fact. As both  $\|\bar{u}\|_\infty$  and  $\|\bar{u} + h\|_\infty$  are bounded by  $\|\bar{u}\|_\infty + \Lambda$ , we can apply Corollary 2.4 and find constants  $\varphi_\bullet, \varphi^\bullet \in D(\beta)$  such that

$$\varphi_\bullet \leq \bar{\varphi} \leq \varphi^\bullet \quad \text{and} \quad \varphi_\bullet \leq \varphi^h \leq \varphi^\bullet \quad \text{a.e. in } Q. \quad (5.2)$$

Now, let us prove (5.1) by writing the problem solved by  $(\zeta^h, \eta^h)$ . We clearly have

$$\partial_t \zeta^h - \Delta \zeta^h + \partial_t \eta^h = 0 \quad \text{a.e. in } Q \quad (5.3)$$

$$\partial_t \eta^h - \Delta \eta^h + \gamma(\varphi^h) - \gamma(\bar{\varphi}) - \gamma'(\bar{\varphi}) \Phi = \zeta^h \quad \text{a.e. in } Q. \quad (5.4)$$

Moreover, both  $\zeta^h$  and  $\eta^h$  satisfy homogeneous initial and boundary conditions (of Dirichlet and Neumann type, respectively). Now, we integrate (5.3) with respect to time and obtain

$$\zeta^h - \Delta(1 * \zeta^h) + \eta^h = 0. \quad (5.5)$$

At this point, we multiply (5.5) and (5.4) by  $\zeta^h$  and  $\eta^h$ , respectively, add the resulting equalities to each other and integrate over  $Q_t$ . The terms involving the product  $\zeta^h \eta^h$  cancel out and we have

$$\begin{aligned} & \int_{Q_t} |\zeta^h|^2 + \frac{1}{2} \int_{\Omega} |(1 * \nabla \zeta^h)(t)|^2 + \frac{1}{2} \int_{\Omega} |\eta^h(t)|^2 + \int_{Q_t} |\nabla \eta^h|^2 \\ &= - \int_{Q_t} (\gamma(\varphi^h) - \gamma(\bar{\varphi}) - \gamma'(\bar{\varphi}) \Phi) \eta^h. \end{aligned} \quad (5.6)$$

Now, for a.a.  $(t, x) \in Q$ , we write the Taylor expansion of  $\gamma$  around  $\bar{\varphi}(t, x)$ . Some function  $\tilde{\varphi}_h$  exists such that

$$\begin{aligned} \gamma(\varphi^h) &= \gamma(\bar{\varphi}) + \gamma'(\bar{\varphi})(\varphi^h - \bar{\varphi}) + \frac{1}{2} \gamma''(\tilde{\varphi}_h)(\varphi^h - \bar{\varphi})^2 \quad \text{a.e. in } Q \\ \min\{\varphi^h, \bar{\varphi}\} &\leq \tilde{\varphi}_h \leq \max\{\varphi^h, \bar{\varphi}\} \quad \text{a.e. in } Q. \end{aligned}$$

Then,  $\varphi_{\bullet} \leq \tilde{\varphi}_h \leq \varphi^{\bullet}$  by (5.2). It follows that  $\gamma''(\tilde{\varphi}_h)$  is bounded since  $D(\beta)$  is an open interval and  $\gamma''$  is continuous. As the same is true for  $\gamma'(\bar{\varphi})$ , we can estimate the right-hand side of (5.6) by accounting for the Young and Hölder inequalities with any  $\delta \in (0, 1)$  as follows

$$\begin{aligned} & - \int_{Q_t} (\gamma(\varphi^h) - \gamma(\bar{\varphi}) - \gamma'(\bar{\varphi}) \Phi) \eta^h \\ &= - \int_{Q_t} (\gamma'(\bar{\varphi}) \eta^h + \frac{1}{2} \gamma''(\tilde{\varphi}_h)(\varphi^h - \bar{\varphi})^2) \eta^h \\ &\leq c \int_{Q_t} |\eta^h|^2 + c \int_{Q_t} |\varphi^h - \bar{\varphi}|^2 |\eta^h| \\ &\leq c \int_{Q_t} |\eta^h|^2 + c \int_0^t \|(\varphi^h - \bar{\varphi})(s)\|_4 \|(\varphi^h - \bar{\varphi})(s)\|_2 \|\eta^h(s)\|_4 ds \\ &\leq c \int_{Q_t} |\eta^h|^2 + \delta \int_0^t \|\eta^h(s)\|_4^2 ds + c_{\delta} \int_0^t \|(\varphi^h - \bar{\varphi})(s)\|_4^2 \|(\varphi^h - \bar{\varphi})(s)\|_2^2 ds. \end{aligned}$$

Now, we recall the Sobolev inequality  $\|v\|_4 \leq C_{\Omega} \|v\|_V$  for every  $v \in V$ , where  $C_{\Omega}$  depends only on  $\Omega$ , and that estimate (2.19) holds for the pair of controls  $\bar{u} + h$  and  $\bar{u}$  and for the corresponding states  $(\vartheta^h, \varphi^h)$  and  $(\bar{\vartheta}, \bar{\varphi})$ . Therefore, we can continue and obtain

$$\begin{aligned} & - \int_{Q_t} (\gamma(\varphi^h) - \gamma(\bar{\varphi}) - \gamma'(\bar{\varphi}) \Phi) \eta^h \\ &\leq c \int_{Q_t} |\eta^h|^2 + \delta C_{\Omega} \int_{Q_t} (|\eta^h|^2 + |\nabla \eta^h|^2) + c_{\delta} \|\varphi^h - \bar{\varphi}\|_{L^2(0,T;V)}^2 \|\varphi^h - \bar{\varphi}\|_{L^{\infty}(0,T;H)}^2 \\ &\leq c \int_{Q_t} |\eta^h|^2 + \delta C_{\Omega} \int_{Q_t} |\nabla \eta^h|^2 + c_{\delta} \|h\|_{L^2(Q)}^4. \end{aligned}$$

At this point, we choose  $\delta$  small enough, rearrange and apply the Gronwall lemma. This yields (5.1).  $\square$



## 6 Necessary optimality conditions

In this section, we derive the optimality condition (2.57) stated in Theorem 2.7. We start from (2.38) and first prove (2.51).

**Proposition 6.1.** *Let  $u^*$  be an optimal control and  $(\varphi^*, \vartheta^*) := \mathcal{S}(u^*)$ . Then, (2.51) holds.*

*Proof.* This is essentially due to the chain rule for Fréchet derivatives, as already said in Section 2, and we just provide some detail. We notice that  $g$  and  $g'$  are computed only at the values of  $\varphi^*$  in (2.51) and we can fix  $\varphi_\bullet, \varphi^\bullet \in D(\beta)$  in order that (2.44) holds for  $\varphi^*$  and modify  $g$  outside of  $[\varphi_\bullet, \varphi^\bullet]$  without changing anything else both in the problem and in the formula we want to prove. Hence, we can assume even  $g'$  to be bounded so that the functional

$$\varphi \mapsto \frac{1}{2} \int_Q (g(\varphi) - \chi)^2$$

is well-defined and Fréchet differentiable in the whole of  $L^2(Q)$ .

It follows that  $\mathcal{F}$  is Fréchet differentiable in  $\mathcal{Z} := L^2(Q) \times L^2(Q)$  and that its Fréchet derivative  $[D\mathcal{F}](\bar{\vartheta}, \bar{\varphi})$  at any point  $(\bar{\vartheta}, \bar{\varphi}) \in \mathcal{Z}$  acts as follows

$$[D\mathcal{F}](\bar{\vartheta}, \bar{\varphi}) : (h_1, h_2) \in \mathcal{Z} \mapsto \int_Q (g(\bar{\varphi}) - \chi) g'(\bar{\varphi}) h_1 + \kappa \int_Q (\bar{\vartheta} - \vartheta_Q) h_2.$$

Therefore, Theorem 5.1 and the chain rule ensure that  $\mathcal{J}$  is Fréchet differentiable at  $u^*$  and that its Fréchet derivative  $[D\mathcal{J}](u^*)$  and any optimal control  $u^*$  acts as follows

$$[D\mathcal{J}](u^*) : h \in \mathcal{X} \mapsto \int_Q (g(\varphi^*) - \chi) g'(\varphi^*) \Phi + \kappa \int_Q (\vartheta^* - \vartheta_Q) \Theta$$

where  $(\Theta, \Phi)$  is the solution to the linearized problem corresponding to  $h$ . Therefore, (2.51) immediately follows from (2.38).  $\square$

The next step is the proof of Theorem 2.6. For convenience, we consider the equivalent forward problem in the unknown  $(\tilde{p}, \tilde{q})$  given by  $(\tilde{p}, \tilde{q})(t) := (p, q)(T - t)$ . However, to simplify notations, we write  $p$  and  $q$  instead of  $\tilde{p}$  and  $\tilde{q}$  in the sequel. Thus, we write the homogeneous initial-boundary value problem

$$\partial_t p - \Delta p - q = f_1 \quad \text{a.e. in } Q \quad (6.1)$$

$$\partial_t q - \Delta q + \alpha q + \partial_t p = f_2 \quad \text{a.e. in } Q \quad (6.2)$$

$$p = 0 \quad \text{and} \quad \partial_n q = 0 \quad \text{a.e. on } \Sigma \quad (6.3)$$

$$p(0) = 0 \quad \text{and} \quad q(0) = 0 \quad \text{a.e. in } \Omega \quad (6.4)$$

with an obvious choice of  $f_1, f_2 \in L^2(Q)$  and  $\alpha \in L^\infty(Q)$ . In order to prove uniqueness, we replace  $f_1$  and  $f_2$  by 0. We multiply the above equations by  $\partial_t p$  and  $q$ , respectively, add the equalities we get to each other and observe that the terms involving the product  $q \partial_t p$  cancel out. Then, we integrate over  $Q_t$  and rearrange. We obtain

$$\int_{Q_t} |\partial_t p|^2 + \frac{1}{2} \int_\Omega |\nabla p(t)|^2 + \frac{1}{2} \int_\Omega |q(t)|^2 + \int_{Q_t} |\nabla q|^2 = - \int_{Q_t} \alpha |q|^2. \quad (6.5)$$



As  $\alpha$  is bounded, we can apply the Gronwall lemma and conclude that  $p = 0$  and  $q = 0$ . As far as existence is concerned, we start deriving the basic formal estimates. As before, we multiply (6.1) by  $\partial_t p$  and (6.2) by  $q$  and perform the same calculation. We obtain an inequality like (6.5) with a different right-hand side, namely

$$- \int_{Q_t} \alpha |q|^2 + \int_{Q_t} f_1 \partial_t p + \int_{Q_t} f_2 q.$$

By owing to the Hölder and Young inequalities, we infer that the above expression is bounded from above by

$$c \int_{Q_t} |q|^2 + \frac{1}{2} \int_{Q_t} |\partial_t p|^2 + c \int_{Q_t} (|f_1|^2 + |f_2|^2).$$

Hence, we have that

$$\|p\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|q\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c \quad (6.6)$$

and the estimate

$$\|\partial_t q\|_{L^2(0,T;H)} \leq c \quad (6.7)$$

immediately follows as a consequence by multiplying (6.2) by  $\partial_t q$ . Therefore, it is clear how to give a rigorous proof based on a Faedo–Galerkin scheme, which provides a sequence  $\{(p_n, q_n)\}$  of approximating solutions obtained by solving just linear systems of ordinary differential equations. Namely, by performing the above estimates on  $(p_n, q_n)$  exactly in the same way and using standard compactness results, one finds a weak limit  $(p, q)$  in the topologies associated to (6.6)–(6.7) and it is immediately clear that  $(p, q)$  is a variational solution of the problem we want to solve. Then, the complete regularity (2.52)–(2.53) and the fact that  $(p, q)$  solves the problem in its strong form follow from the general theory. So, Theorem 2.6 actually holds.  $\square$

At this point, we are ready to prove Theorem 2.7 on optimality, i.e., the necessary condition (2.57) for  $u^*$  to be an optimal control in terms of the solution  $(p, q)$  of the adjoint problem (2.54)–(2.56). So, we fix an arbitrary  $u \in \mathcal{U}_{ad}$  and write the variational formulations of both the linearized problem (corresponding to  $h = u - u^*$ ) and the adjoint problem.

The equations become

$$\int_Q \partial_t \Theta v + \int_Q \nabla \Theta \cdot \nabla v + \int_Q \partial_t \Phi v = \int_Q m(u - u^*) v \quad (6.8)$$

$$\int_Q \partial_t \Phi v + \int_Q \nabla \Phi \cdot \nabla v + \int_Q \gamma'(\varphi^*) \Phi v = \int_Q \Theta v \quad (6.9)$$

$$- \int_Q \partial_t p v + \int_Q \nabla p \cdot \nabla v - \int_Q q v = \kappa \int_Q (\vartheta^* - \vartheta_Q) v \quad (6.10)$$

$$\begin{aligned} & - \int_Q \partial_t q v + \int_Q \nabla q \cdot \nabla v + \int_Q \gamma'(\varphi^*) q v - \int_Q \partial_t p v \\ & = \int_Q (g(\varphi^*) - \chi) g'(\varphi^*) v \end{aligned} \quad (6.11)$$

where (6.8) and (6.10) have to hold for every  $v \in L^2(0, T; V_0)$ , while (6.9) and (6.11) are required for every  $v \in L^2(0, T; V)$ . In particular (6.9) and (6.11) also contain the homogeneous Neumann conditions for  $\Phi$  and  $q$ . Moreover,  $\Theta$  and  $p$  have to satisfy the homogeneous Dirichlet boundary conditions, in addition. Finally, the functions at hand satisfy the homogeneous initial or final conditions as specified in (2.49) and (2.56). We choose  $v = p$ ,  $v = q$ ,  $v = -\Theta$  and  $v = -\Phi$  in (6.8)–(6.11), respectively, and add all the equality we obtain to each other. The most part of the terms cancel out and we obtain

$$\int_Q \partial_t (\Theta p + \Phi q + \Phi p) = \int_Q \{m(u - u^*)p - \kappa(\vartheta^* - \vartheta_Q)\Theta - (g(\varphi^*) - \chi)g'(\varphi^*)\Phi\}.$$

Due to the initial and final conditions, the left-hand side vanishes and we deduce that

$$\int_Q \{\kappa(\vartheta^* - \vartheta_Q)\Theta + (g(\varphi^*) - \chi)g'(\varphi^*)\Phi\} = \int_Q m(u - u^*)p.$$

As the left-hand side is  $\geq 0$  by (2.51), it follows that the same is true for the right-hand side. As  $u \in \mathcal{U}_{ad}$  is arbitrary, this implies the pointwise inequality (2.57) and the proof of Theorem 2.7 is complete.  $\square$

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